XII. $BC=a : EF=x : DF=y : DE=z$

$:: AC=b : AF=v : EF=x : AE=w$

$:: AB=c : AE=w : ED=z : AD=v+y.$

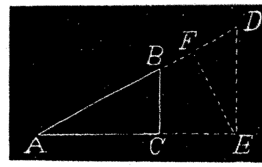

Fig. 8.

The above condensed form is self-explanatory, as are also the two following.

We leave the selection of simple proportions, the derivation and solution of consequent equations, as an exercise for the interested reader.

XIII. $BC=a : DE=x : DL=y,$

$:: AB=b : AE=z : LF=FE=v,$

$:: AB=c : AD=v+y : DF=x-v.$

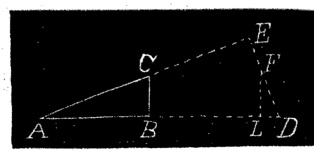

Fig. 9.

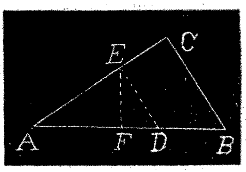

Fig. 10.

XIV. $BC=a : ED=EC=x : FD=y : EF=z,$

$:: AC=b : AE=b-x : EF=z : AF=v,$

$:: AB=c : AD=v+y : ED=x : AE=b-x.$

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from May Number.]

Each of the regular groups of degree six contains only one subgroup of the type $(abc.def)$. Since no substitution of the form $abcde$ can transform this into itself the group of order 30 is impossible.

If a group of order 60 exists it must contain six subgroups of the type $(abcde)_{10}$. We may assume that it contains $(abcde)_{10} \equiv G_f$. G_b must then contain $ac.de$ and a substitution of the type $abcde$ which contains the letters a, c, d, e, f. We may assume that this substitution is $acd_1e_1f_1$. It is then necessary that $ac.de.acd_1e_1f_1 = af_1e_1d_1c.ac.de$. Hence

$$acd_1e_1f_1 = acdfe \text{ or } acefd.$$

Since $acefd.adbec=bef$ every group of order 60 must contain

$$(abcde)_{10} \text{ and } acdfe.$$

These substitutions generate a group whose order $\overline{>}60$, hence only one group of order 60 is possible.

We shall prove that these substitutions generate a group of order 60 by employing a very elementary but somewhat lengthy method. Representing the substitutions of $(abcde)_{10}\equiv 1$, $abcde$, $acebd$, $adbec$, $aedcb$, $ab.ce$, $ac.de$, $ad.bc$, $ae.bd$, $be.cd$ respectively by $1=s_1$, s_2, s_3, s_4, s_5, c_6, s_7, s_8, s_9, s_{10} and $acdfe$ by t, we form the rectangle

$$\begin{array}{llll}
1 & s_2 & s_3 & \ldots\ldots\ldots\ldots s_{10} \\
t & s_2 t & s_3 t & \ldots\ldots\ldots\ldots s_{10} t \\
t^2 & s_2 t^2 & s_3 t^2 & \ldots\ldots\ldots\ldots s_{10} t^2 \\
t^3 & s_2 t^3 & s_3 t^3 & \ldots\ldots\ldots\ldots s_{10} t^3 \\
t^4 & s_2 t^4 & s_3 t^4 & \ldots\ldots\ldots\ldots s_{10} t^4 \\
t_1 & s_2 t_1 & s_3 t_1 & \ldots\ldots\ldots\ldots s_{10} t_1
\end{array}$$

Where t_1 is any substitution generated by $(abcde)_{10}$ and $acdfe$ which is not found in the preceding five rows. These substitutions are all different. They form a group if $t_1{}^2$ is contained in the first five rows and

$$t^{\alpha} s_{\beta} = s_{\gamma} t^{\delta} \text{ or } s_{\gamma} t_1, \; t_1 s_{\beta} = s_{\gamma} t^{\delta} \text{ or } s_{\gamma} t_1$$

$$(\beta, \gamma = 1, 2, \ldots\ldots 10), \; (\alpha, \delta = 1, 2, \ldots\ldots 4).$$

Instead of allowing β to have 10 values it is clearly sufficient to assign to it only the two values of 2 and 6 since $abcde$ and $ab.ce$ generate $(abcde)_{10}$. The following shows that the necessary conditions are fulfilled:

$$\begin{array}{ll}
ts_2 = adf.bce = s_{10} t^2 & ts_6 = aeb.cdf = s_3 t^3 \\
t^2 s_2 = aed.bcf = t_1{}^* & t^2 s_6 = adcfb = s_9 t_1 \\
t^3 s_2 = afdbc = s_9 t^4 & t^3 s_6 = afedb = s_4 t \\
t^4 s_2 = bc.ef = s_5 t & t^4 s_6 = acb.def = s_8 t^4 \\
t_1{}^2 = ade.bfc = s_5 t^3 & t_1 s_2 = bd.cf = s_4 t^3 \\
 & t_1 s_3 = acf.bed = s_9 t^2
\end{array}$$

There is therefore one group of order 60, viz :

(1) $$(abcde)_{10}(acdfe) = (abcdef)_{60}.$$

If there is a primitive group of order 120 it may be assumed that it con-

*In the above rectangle f is followed by the same letter as in the corresponding t or t_1. Since it is not followed by b in t, $aed.bcf$ cannot be contained in the first five lines and may therefore be used for t. All these relations may be readily found if this property is observed.

tains $(abcde)_{20}$ and therefore $(abcdef)_{60}$. Since half of its substitutions must be negative it must contain $(abcdef)_{60}$ as a self-conjugate subgroup.

The order of a group which satisfies these conditions cannot be less than 120. From this we see that there cannot be more than one group of this order. That there is one follows from the facts that $acbe$ belongs to $(abcde)_{20}$ and transforms $acdfe$ into $acbdf=(s_9t_1)^2=$some substitution of $(abcdef)_{60}$.

The other primitive groups of degree six must contain subgroups of degree five which contain substitutions of one of the two types

$$ab \qquad abc$$

They must therefore be the alternating and the symmetric group. The following is therefore a complete list of these groups :

Order	Group
60	$(abcdef)_{60}$
120	$(abcdef)_{120}$
360	$(abcdef)$pos
720	$(abcdef)all$

Remarks.

We have now finished the explanations of the elementary methods of group construction. By means of these we have been able to find, with a reasonable amount of labor, all the groups whose degree does not exceed six. It scarcely needs to be stated that this labor could have been considerably reduced by employing more advanced methods. In fact, we did not endeavor so much to find these groups by the least labor as to find them in such a way as to illustrate some of the most important elementary methods of group construction.

We are indebted to our honored teacher, Professor F. N. Cole not only for many of these methods but also for the fundamental ideas.

Most of the theorems that we have developed are found in Part I of Netto's Theory of Substitutions (American Edition). In some instances it seemed desirable to change the method of proof either because we had not yet developed the principles upon which Netto's proof is based, or because we desired to call attention to some special property. In a few instances our purposes required us to pursue the demonstration farther than is done by this author.

We did not enter into a special study of methods of operating with substitutions. Some of the more important ones have been incidentally explained. For further explanations we would refer to Senet's Algèbre Supérieure, Part IV, (this part is found in the second volume of this work), and to Part I of Netto.

In these works is also found considerable on the analysis of a substitution. The first 15 pages of the first volume of Gordan's Invariantentheorie contain considerable on this point. For the more advanced methods of operation we have

to refer to the classical work on this subject, Jordan's Traité des Substitutions, and to the periodicals.

Before entering upon the development of more advanced methods of group construction we shall study some of the relations which exist between substitution groups and functions containing a finite number of letters. These relations will not only show how substitution groups may be utilized but they may also serve as a means of arriving at important properties of substitution groups.

Leipzig, Germany, September 20, 1895.

SIMULTANEOUS QUADRATIC EQUATIONS.

By I. H. BRYANT, M. A., Instructor of Mathematics, Waco High School, Waco, Texas.

[Continued from May Number.]

The discussion in this article is restricted to two unknown quantities. Cases 1, 2, 4, and 5 apply to two variables just as they are stated in the previous article. In Case 3, the restriction that each factor must occur twice is unnecessary when only two variables occur. It is sufficient for one factor to occur in each equation. This reduces Case 3 to Case 2. For two variables, Cases 6, 7, and 8 become one and the same, as no restrictions are necessary.

The following Cases are applicable to two variables only. Express the equations thus for Cases 9 and 10:

$$ax^2+by^2+cxy+dx+ey+f=0. \qquad 1.$$

$$a'x^2+b'y^2+c'xy+d'x+e'y+f'=0. \qquad 2.$$

CASE 9. When $a : a' :: c : c' :: d : d'$. If this is true, it is obvious that the terms containing x can be eliminated. This holds true when the terms of any one, or any two, of the three ratios are zero.

CASE 10. When $a : a' :: b : b' :: d^2 : d'^2 :: e^2 : e'^2$, and when $d : d' :: e : e'$.

By alternation, $\frac{e}{d}=\frac{e'}{d'}$, $\frac{b}{a}=\frac{b'}{a'}$.

Let $\frac{e}{d}=r$. Then $\frac{e'}{d'}=r$, $\frac{b}{a}=r^2$, $\frac{b'}{a'}=r^2$.